Let us consider the dependence of the wavelength λ on the Rayleigh number, and also on the amplitude and frequency of the exciting oscillations. Calculation shows that, in the above-critical region ($Ra > Ra^*$), λ is practically independent of Ra and α (Fig. 3b). Hence the phase velocity of TCW propagation is also independent of Ra and α and is only a function of ω. In fact, increase in ω leads to decrease in λ; the decrease is most significant for small ω ($\omega < 3$) and for large frequencies the dependence of λ on ω markedly decreased (Fig. 3a).

In conclusion, it should be noted that the results on TCW obtained in the present work by numerical calculation are in good qualitative agreement with those of analysis [3, 4] and of physical experiments [5, 6].

NOTATION

$Gr = \beta g d^3(|\gamma d| + A_0)/\nu^2$, Grashof number; $Ra = Gr\alpha Pr = \beta g d^3 \gamma d/\nu a$, Rayleigh number; $\alpha = \gamma d/(|\gamma d| + A_0)$, parameter characterizing the relation between the vertical temperature drop in the layer and the amplitude of the temperature oscillations at the wall; $\gamma = (T_1 - T_2)/d$, vertical temperature gradient in layer; l, length of layer; d, layer thickness; A_0, maximum amplitude of temperature oscillations at wall; ν, kinematic viscosity; a, thermal conductivity; $Pr = \nu/a$, Prandtl number; β, coefficient of thermal compressibility; g, acceleration due to gravity; $\Theta(x,y,t)$, dimensionless temperature in layer; Ra^*, critical Rayleigh number corresponding to loss of mechanical equilibrium of the layer; ν/d, scale of velocity; d^2/ν, scale of time; ω, frequency of exciting oscillations; L, depth of penetration, defined as the distance from the side wall at which the amplitude is reduced by a factor of 10; $A_\Theta(x) = 2(1-|\alpha|)^{-1}[\max\Theta(x;\ 0.5;\ t) - \min\Theta(x;\ 0.5;\ t)]$, amplitude of temperature oscillations in median line of cavity $y = 0.5$, $t \in [t_1, t_2]$, $t_2 - t_1 = 2\pi/\omega$.

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APPROXIMATE SOLUTION OF EXTENDED GRAETZ PROBLEM BY ORTHOGONAL COLLOCATION

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UDC 536.242

The method of orthogonal collocation is applied to the Graetz problem. The method allows a very accurate solution to be obtained in the initial region, where the Fourier series converges very slowly.

1. Introduction

Linear partial differential equations (LPDE) are the mathematical models most commonly used to describe engineering systems. Boundary-value problems for these equations may be solved by means of Fourier

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TABLE 1. Eigenvalues of Classical Graetz Problem ($Pe \to \infty$) in Comparison with Eigenvalues $\underline{\underline{Q}}_+$ for $N = 12$

k	1	2	3	4	5	6
Fourier series	7,318	41,61	113,9	215,2	348,2	513,9
12-point collocation	Difference between Fourier series and 12-point collocation					
	$4\cdot10^{-12}$	$5\cdot10^{-11}$	$3\cdot10^{-8}$	$2\cdot10^{-4}$	0,2	10,5
k	7	8	9	10	11	12
Fourier series	711,2	940,5	1201	1495	1820	2178
12-point collocation	Eigenvalues					
	827,7	1539	3627	12243	81559	$4,1\cdot10^{6}$

series. For example, in [2] the first 20 eigenvalues and the corresponding eigenfunctions for an extended Graetz problem were obtained by the Runge—Kutta method.

However, it is not at all obvious that the classical Fourier-series method allows the basis function that is best in terms of the rate of convergence to be chosen. In [3], for example, it was shown that to achieve the same degree of accuracy in approximating the solution of the heat-conduction equation requires many fewer terms of a polynomial basis function than for a trigonometric Fourier series.

From experience in computation it is known that the approximation to the solution of LPDE obtained by means of finite Fourier series is good for large values of the independent variables but poor for small values.

By the orthogonal-collocation method, the solution of LPDE is obtained as an expansion in a finite number of basis functions, usually Legendre polynomials. The solution reduces to finding the eigenvalues of a certain matrix. The first eigenvalue of this matrix is usually extremely close to the eigenvalue of the differential operator, but larger eigenvalues differ more strongly. However, it is found that the resulting solution is more accurate than that obtained using Fourier series, especially for small values of the independent variables [3,5,6].

We shall attempt to find an explanation for this difference in accuracy.

2. Extended Graetz Problem

It is assumed that an infinite tube of radius R has a wall temperature T_0 on the section from $z = 0$ to $z \to \infty$. At $z \to -\infty$, a Newtonian liquid is admitted to the tube with temperature T_b. It flows along the tube in laminar conditions with linear velocity v_z and at $z \to \infty$ it acquires a temperature T_0. The differential equations have the form

$$v_z \frac{\partial T}{\partial z} = 2 \langle v_z \rangle \left(1 - \frac{r^2}{R^2}\right) \frac{\partial T}{\partial z} = \frac{k}{\rho c_p} \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r}\right) + \frac{\partial^2 T}{\partial z^2}\right], \tag{1}$$

where z is the longitudinal coordinate and r is the radial coordinate; $T(r, z)$ is the temperature at the point (r, z); $\langle v_z \rangle$ is the mean axial flow velocity; k is the thermal conductivity; c_p is the specific heat; and ρ is the density of the liquid.

We introduce the following dimensionless variables:

$$x = \frac{r}{R}, \quad y = \frac{kz}{2\rho c_p \langle v_z \rangle R^2} = \frac{z}{\mathrm{Pe}\, R}; \quad \theta = \frac{T - T_0}{T_b - T_0},$$

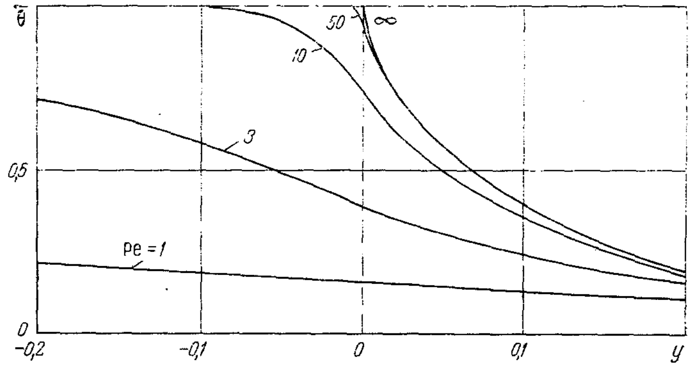

Fig. 1. Flow-core temperature $\bar{\theta}$ – Eq. (5) – as a function of $y = z/(RPe)$ for various Pe.

and transform Eq. (1) to obtain

$$(1 - x^2) \frac{\partial \theta}{\partial y} = \frac{1}{x} \frac{\partial}{\partial x} \left(x \frac{\partial \theta}{\partial x} \right) + \frac{1}{\mathrm{Pe}^2} \frac{\partial^2 \theta}{\partial y^2}. \tag{2}$$

The boundary conditions for Eq. (2) are as follows:

$$\theta(1, -\infty) = 1, \quad \left.\frac{\partial \theta}{\partial x}\right|_{x=0} = 0,$$
$$\left.\frac{\partial \theta}{\partial x}\right|_{x=1, y<0} = 0, \quad \theta(1, y \geqslant 0) = 0. \tag{3}$$

The last term in Eq. (2) may usually be neglected, since for Pe > 30–50 its effect is very small everywhere except in the region close to y = 0 and that close to the wall, where the convective term vanishes; this is shown, for example, in [1].

The eigenfunctions $F_k(x)$ of Eq. (2) are solutions of the equation

$$\frac{1}{x} \frac{d}{dx} \left(x \frac{dF}{dx} \right) - \left[(1 - x^2) \lambda - \frac{\lambda^2}{\mathrm{Pe}^2} \right] F = 0 \tag{4}$$

with boundary conditions

$$\left.\frac{\partial F}{\partial x}\right|_{x=0} = 0, \quad \left.\frac{\partial F}{\partial x}\right|_{x=1} = 0 \quad \text{for} \quad y < 0,$$
$$F(1) = 0 \quad \text{for} \quad y \geqslant 0.$$

The solution of Eq. (2) is written in the form of an infinite series

$$\theta = \sum_{0}^{\infty} A_k F_k(x) \exp(\lambda_k y),$$

where A_k are the coefficients of the Fourier series.

Since the boundary conditions at x = 1 are different for y < 0 and y ≥ 0, the eigenvalues and eigenfunctions likewise should be found for each region: y < 0 (λ_-, F_-) and y ≥ 0 (λ_+, F_+). They may be determined by numerical integration of Eq. (4) and a selection of λ_k such as to satisfy the boundary conditions for Eq. (4). Since Eq. (4) is not a Sturm–Liouville problem, the coefficients A_{k+} and A_{k-} cannot be determined using the usual simple formulas but only numerically. A solution was obtained by this means in [2]. However, as will be shown below, for small Pe around y = 0 this approach may give results that are even qualitatively incorrect.

In our analysis, the following quantities will be used:

a) the flow-core temperature $\bar{T}(z)$,

$$\int_A \rho c_p v_z T d_A = \bar{T} \int_A v_z \rho c_p d_A,$$

$$\bar{\theta} = \frac{\bar{T} - T_0}{T_b - T_0} = 4 \int_0^1 x(1 - x^2)\,\theta dx; \tag{5}$$

b) the Nusselt number

$$\mathrm{Nu} = \frac{2Rh}{k} = \frac{-2 \left.\frac{\partial T}{\partial x}\right|_{x=1}}{\bar{T} - T_0} = -\frac{2 \left.\frac{\partial \theta}{\partial x}\right|_{x=1}}{\bar{\theta}}; \tag{6}$$

c) the total heat transfer from z = 0 to the wall

$$J_z = \int_0^z -k \left.\frac{\partial T}{\partial r}\right|_{r=R} 2\pi R dz$$

or

$$J = \frac{J_z}{J_\infty} = \frac{J_z}{\pi R^2 \langle v_z \rangle \rho c_p (T_b - T_0)} = -4 \int_0^y \left.\frac{\partial \theta}{\partial x}\right|_{x=1} dy. \tag{7}$$

As Pe → ∞, we have the limiting values

$$\bar{\theta}(0) = 1; \quad J = 1 - \bar{\theta}. \tag{8}$$

3. Numerical Solution

We find the numerical solution of Eqs. (2) and (3) by the orthogonal-collocation method. We define the following vectors: $\{x_i\} = \underline{x}$ is the vector of the collocation-point abscissa; $\{u_i\} = \{x_i^2\} = \underline{u}$ is the vector of the zero of the displacement Legendre polynomial $P_N(u)$ of degree N;

$$\{\theta_i\} = \theta(x_i, y) = \underline{\theta}(y), \quad \{\varphi_i\} = \frac{\partial \theta}{\partial y}(x_i, y) = \underline{\varphi}(y).$$

We also construct the diagonal matrices $\underline{\underline{U}} = \{u_{ii}\}$ and $\underline{\underline{V}} = \{1 - u_{ii}\}$. Using this notation, the following collocation equation may be obtained:

$$\frac{d\underline{\theta}}{dy} = \underline{\varphi}; \quad \frac{d\underline{\varphi}}{dy} = \mathrm{Pe}^2 \underline{\underline{V}}\,\underline{\varphi} - 4\mathrm{Pe}^2 (\underline{\underline{U}}\,\underline{\underline{B}} + \underline{\underline{A}})\,\underline{\theta}. \tag{9}$$

The matrices $\underline{\underline{A}}$ and $\underline{\underline{B}}$ for $d\theta/dy$ and $d^2\theta/du^2$ are constructed similarly. An approximation for θ is obtained by the Lagrange interpolation formula, using N collocation points together with the point u = 1 and, in some cases, also the point u = 0. Thus, for N + 2 points, we have the following approximation for θ:

$$\theta(u) = \sum_{j=0}^{N+1} \frac{p(u)}{(u - u_j)\, p^{(1)}(u_j)}\, \theta(u_j) = \sum_{j=0}^{N+1} P_j(u)\, \theta_j,$$

where

$$p(u) = u(u - u_1) \ldots (u - u_n)(u - 1)$$

$$p^{(1)}(u_j) = \left.\frac{dp}{du}\right|_{u=u_j};$$

$$\left.\frac{d\theta}{du}\right|_{u=u_i} = \sum_{j=0}^{N+1} \left.\frac{dP_j(u)}{du}\right|_{u=u_i} \theta_j = \sum_{j=0}^{N+1} P_j^{(1)}(u_i)\,\theta_j = \sum_{j=0}^{N+1} A_{ij}\theta_j;$$

$$\left.\frac{d^2\theta}{du^2}\right|_{u=u_i} = \sum_{j=0}^{N+1} P_j^{(2)}(u_i)\,\theta_j = \sum_{j=0}^{N+1} B_{ij}\theta_j.$$

At each of the collocation points u_i the first and second derivatives of θ are calculated as the weighted sum of N + 2 ordinates: $\theta(0), \ldots, \theta(u_i), \ldots, \theta(1)$. The formula is especially convenient when $\theta(0)$ and $\theta(1)$ are given. In our case, both $\theta(0)$ and $\theta(1)$ are unknown for $y < 0$, while $\theta(1) = 0$ for $y \geq 0$. Therefore, it is necessary to use an interpolation polynomial of order N + 1: $p(u) = (u - u_1) \ldots (u - u_N)(u - 1)$.

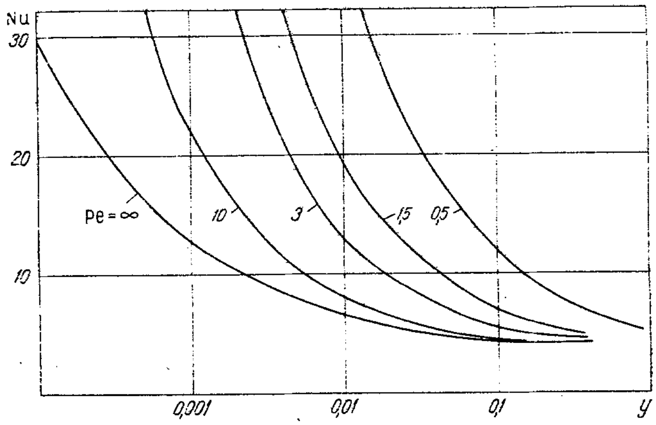

Fig. 2. Nu — Eq. (6) — as a function of y for various Pe.

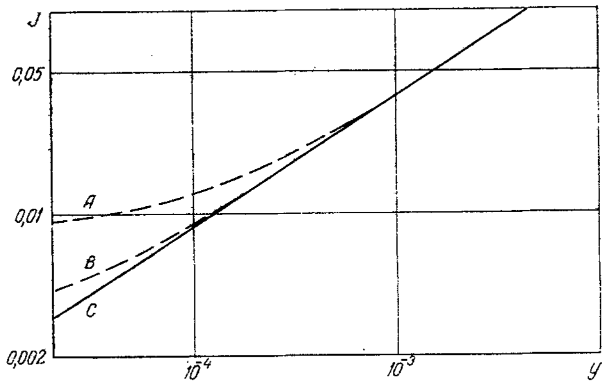

Fig. 3. Dependence of heat flow J — Eq. (7) — on y as Pe → ∞. Curve C shows the results given by 12-point collocation and by Eq. (12), while A and B correspond to Fourier series with 12 and 30 terms.

For the variable $u = x^2$, the Laplace operator in Eq. (2) takes the form

$$\frac{1}{x}\frac{d}{dx}\left(x\frac{d}{dx}\right) = 4\left(u\frac{d^2}{du^2} + \frac{d}{du}\right).$$

Substitution of the collocation vectors $\underline{u}$, $\underline{\theta}$, $\underline{\varphi}$ and the matrices $\underline{\underline{U}}$, $\underline{\underline{V}}$, $\underline{\underline{A}} = \{A_{ij}\}$, $\underline{\underline{B}} = \{B_{ij}\}$ in Eq. (2) also leads to the system of 2N ordinary differential equations with constant coefficients in Eq. (9).

For $y \geq 0$ and $\theta(1) = 0$, the given calculation scheme can be used directly. For $y < 0$, an additional (2N + 1)-th equation is formulated, corresponding to the boundary condition $\partial\theta/\partial u|_{u=1, y<0} = 0$:

$$\frac{d\theta}{du}\Big|_{u=1, y<0} \approx \sum_{j=1}^{N+1} A_{N+1, j}\theta_j = 0.$$

The solution of Eq. (9) is sought in the form

$$\frac{d\underline{\Psi}}{dy} = \underline{\underline{Q}}\underline{\Psi}, \quad \underline{\Psi} = (\theta_1, \ldots, \theta_n, \varphi_1, \ldots, \varphi_N),$$

$$\underline{\underline{Q}} = \begin{pmatrix} \underline{\underline{0}} & \underline{\underline{1}} \\ -4\mathrm{Pe}^2(\underline{\underline{U}}\,\underline{\underline{B}} + \underline{\underline{A}}) & \mathrm{Pe}^2\underline{\underline{V}} \end{pmatrix} = \begin{cases} \underline{\underline{Q}}_+ & \text{for } y \geqslant 0, \\ \underline{\underline{Q}}_- & \text{for } y < 0, \end{cases} \tag{10}$$

$$\underline{\Psi} = \exp(\underline{\underline{Q}}y)\,\underline{\Psi}_0 = \underline{\underline{S}}\exp(\underline{\underline{\Lambda}}y)\,\underline{\underline{S}}^{-1}\underline{\Psi}_0. \tag{11}$$

The diagonal matrix $\underline{\underline{\Lambda}}$ contains 2N eigenvalues $\underline{\underline{Q}}$. Of the eigenvalues $\underline{\underline{Q}}_+$, N are positive and N are negative. Since Ψ should be finite for all $y \geq 0$, the series of matrices $\underline{\underline{S}}_+^{-1}$ which corresponds to positive eigenvalues should be orthogonal to the vector $\underline{\Psi}_0$. Of the eigenvalues $\underline{\underline{Q}}_-$, N − 1 are negative, N are positive, and one is equal to zero. The series $\underline{\underline{S}}_-^{-1}$ which corresponds to negative eigenvalues should be orthogonal to Ψ_0 for the same reason. Thus we have 2N − 1 equations for the components of the vector Ψ_0. The 2N-th equation follows from the condition $\theta(y) \to 1$ as $y \to \infty$.

The general solution $\bar{\theta}(y)$ is now obtained by the diagonalization of the two 2N × 2N matrices $\underline{\underline{Q}}_+$ and $\underline{\underline{Q}}_-$ and the subsequent solution of 2N linear algebraic equations by the method of Gauss exclusion. The scalar quantity in Eq. (5) is obtained by Gauss quadrature over the collocation points of the vector $\underline{\theta}(y)$.

4. Discussion of Results

Table 1 presents a comparison of the eigenvalues λ_k of the matrix $\underline{\underline{Q}}_+$ for Pe → ∞ and for N = 12 with the eigenvalues of the differential operator. The first two values are equal, within the accuracy of the calculation. This means that the solutions $\underline{\theta}(y)$ or $\bar{\theta}(y)$ are good approximations to the accurate solution for "large" y.

As is evident in Fig. 1, the function $\bar{\theta}(y)$ is practically independent of Pe for Pe > 50, except for a small region in the vicinity of y = 0. In Fig. 2, the function Nu(y) [6] is shown for various Pe. A similar figure is shown in [2], but there the curves reached a finite value as y → 0 for Pe < 10, which contradicts the experimentally observed dependence $Nu \approx (yPe)^{-1/2}$ for Pe < 10 [12]. Our curves agree with this result.

In [13] the following empirical relation was found:

$$J(\mathrm{Pe} \to \infty) = 1 - \bar{\theta} = 4.0698 y^{2/3}, \tag{12}$$

which is true with accuracy 5% for $y < 10^{-3}$.

In Fig. 3 curves of J(y) calculated using 12 and 30 terms of the Fourier series are shown (dashed curves) together with a curve calculated by 12-point collocation and the curve given by Eq. (12) (the last two curves coincide and are shown by the solid curve in Fig. 3). From Fig. 3 it is evident that the finite Fourier series does not correspond to Eq. (12), while the polynomial series is very close to Eq. (12) beyond $y = 10^{-5}$. In the limit as y → 0, of course, the curves diverge, since the infinite derivative $\partial\theta/\partial x|_{x=1, y=0}$ cannot be represented as a polynomial. It is found that, at $y = 2 \cdot 10^{-5}$, the accuracy obtained in 12-point collocation would require more than 100 terms of the Fourier series to be taken into account.

This may be a result of the rapid increase in the eigenvalues of the matrix $\underline{\underline{Q}}_+$ (see Table 1). The condition for a series of N terms to give satisfactory accuracy is that $\lambda_N y \gg 1$. This condition is better satisfied by the collocation series ($\lambda_{12} = 4.1 \cdot 10^6$) than by the Fourier series ($\lambda_{12} = 2178$).

Thus, in a number of cases, the proposed method gives results for the solution of a boundary-value problem that are better from both a physical and a mathematical point of view than those obtained using Fourier series.

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